Gumbel Distribution

Álaze Gabriel do Breviário
Specialist in Finance and Controllership (USP-2023). Specialist in Financial Management (UNINTER-2022). Specialist in Teaching and Research for Higher Education (2015). Specialist in Finance and Controllership (2014). Graduated in Business Management (2012). Studying undergraduate degree in Accounting Sciences UNIMES (6th period). Studying undergraduate degree in Statistics UFSCar (4th period). Pursuing a postgraduate degree in Constitutional and Administrative Law. Bachelor's and Master's Degree in Theology, Free Ecclesiastical Courses at the University of the Bible (2015). Student-researcher on: networks; Fields; Institutions; Organizations; economic sociology; financialization; scientific paradigms; organizational theories; scientific research methodology; teaching in higher education. Lyric poet (poems, acrostics), narrator (novels, chronicles, biographies) and playwright (farces, tragedies). Blog administrator. Professional writer. Contestant. He has very fragmented and diversified academic and professional trajectories, with a concentration in the administrative and accounting areas. She participated in the extension course Equity in access to graduate studies for underrepresented populations to master's courses, at the Federal University of São Carlos - UFSCar.
E-mail: alaze_p7sd8sin5@yahoo.com.br

ABSTRACT
This work seeks to reflect on the Gumbel Distribution. It synthesizes the space-time lapse of its advent, its utilities, characteristics, relations with other distributions and estimators. For this, a bibliographic review of classical authors of statistical inference and of documents made available on the websites of Departments of Exact Sciences of some Brazilian universities is carried out. It is argued that such a distribution is suitable for modeling the behavior of earthquakes, floods and other natural disasters, as well as financial investments and hydrology in general. Therefore, it is very important in scientific research on such phenomena.

Keywords: Gumbel, Gumbel distribution, Distribution of Extreme Value.

1 INTRODUCTION

Emil Julius Gumbel was born on July 18, 1891, in Munich, Germany, and died on September 10, 1966, at the age of 75, in New York City. He was a German pacifist mathematician and writer born in Munich, a socialist and opponent of the Nazi regime and one of the historical German pacifist leaders, who as a mathematician was one of the creators *of the theory of extreme value*. Descended from a traditional Jewish family from Württemberg, he was the first of three children of the well-to-do Munich merchant couple **Hermann** and **Flora Gumbel**, and was educated in his hometown. At the Kaiser-Wilhelm-Gymnasium he received a humanistic education, graduated in statistics at the University of Munich (1913), where he also received his Ph.D (1914), shortly before the outbreak of World War I. He lived in Berlin (1916-1932), where he distinguished himself in intellectual circles as a writer and pacifist. He became Professor of Mathematical Statistics at the University of Heidelberg, initially *(1923) as an external Privatdozent* and then as *Außerordentlicher Professor* (1924) (UFCG, 2017; USP, 2017; PA, 2017; CPRM, 2017; FRANCO *et all*, 2014; MOOD; GRAYBILL; BOES, 1974; MORETTIN; BUSSAB, 2010; TOMAZ, 2008; WATANABE, 2013).

After the murder of a friend, he investigated several political assassinations and published his findings in Vier Jahre politischer Mord (1922) and *Die Denkschrift des Reichsjustizministers über 'Vier Jahre politischer Mord' (1924*). He married (1930) a German divorcee, **Marie Luise Czettritz**, the daughter of a general. He was one of the 33 signatories of the political manifesto *Dringender Appell* (1932) and then had his teaching license revoked by the university's ultra-conservative board dominated by Nazi youth. With the definitive rise of the Nazis to power (1933), he went into exile in France, where he was hired by the University of Lyon (1934). With **Leonard Tippett and** Ronald Fisher**, he formulated the mathematical field of** extreme value theory *and created what he called the* Gumbel distribution (1935) (UFCG, 2017; USP, 2017; PA, 2017; CPRM, 2017; FRANCO *et all*, 2014; MOOD; GRAYBILL; BOES, 1974; MORETTIN; BUSSAB, 2010; TOMAZ, 2008; WATANABE, 2013).

Extreme value theory is a branch of statistics that deals with extreme deviations from the average number of probability distributions. The general theory aims to evaluate the type of probability distributions generated by processes. Extreme value theory is important for assessing the risk of highly exceptional events, such as 100-year floods, which are of great value in hydrological studies. Naturalized French (1939) and living in Marseille, with the Tedesca invasion of that country, he fled to Portugal and emigrated to the United States (1940) where he became a naturalized (1945) and distinguished himself as an adjunct professor at the University of Colombia (1953-1966). He published many articles on the subject, such as *Les valeurs extrêmes des distribuições statistiques* (1935), Statistik of Extremes *(1958), Distributions del valeurs extremes en plusieurs dimensions* (1960), Bivariate Logistic Distributions (1961) and Some Analytical Properties of Bivariate Extreme *Distributions (1967),* the latter with **C. K. Mustafi**, and died in New York, aged 75. He also published works on politics such as *Verschwörer. Zur Geschichte und Soziologie der deutschen nationalistischen Geheimbünde 1918 - 1924 (*1924) (UFCG, 2017; USP, 2017; PA, 2017; CPRM, 2017; FRANCO *et all*, 2014; MOOD; GRAYBILL; BOES, 1974; MORETTIN; BUSSAB, 2010; TOMAZ, 2008; WATANABE, 2013).

When he died, Gumbel's papers were part of the *Emil J. Gumbel Collection, Political Papers of an Anti-Nazi Scholar in Weimar and Exile.* These documents include microfilm reels documenting their activities against the Nazis (UFCG, 2017; USP, 2017; PA, 2017; CPRM, 2017; FRANCO *et all*, 2014; MOOD; GRAYBILL; BOES, 1974; MORETTIN; BUSSAB, 2010; TOMAZ, 2008; WATANABE, 2013).

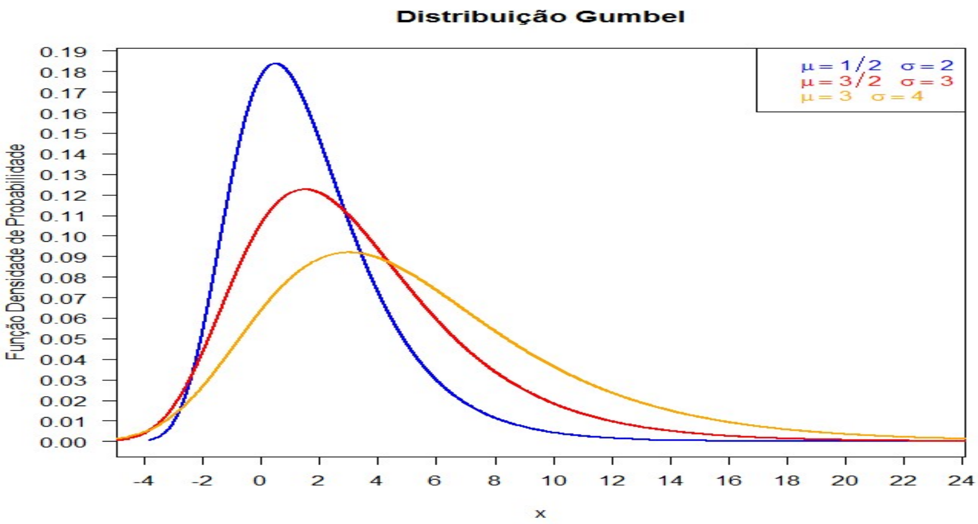
2 CHARACTERIZATION OF THE GUMBEL DISTRIBUTION

The Gumbel distribution is Type I of the extreme value distribution family, which is the one that contains limit distributions for extreme values in a random sample, when the sample size grows (UFCG,

2017; USP, 2017; PA, 2017; CPRM, 2017; FRANCO *et all*, 2014; MOOD; GRAYBILL; BOES, 1974; MORETTIN; BUSSAB, 2010; TOMAZ, 2008; WATANABE, 2013). A variable Y has Gumbel distribution if it has probability density function given by:

$$f(x) = \frac{1}{\sigma} \exp\left[\frac{x-\mu}{\sigma} - \exp\left(\frac{x-\mu}{\sigma}\right)\right] \qquad x \in (-\infty, \infty)$$

in which $\sigma = 1/\delta$it is $\mu = \log(\alpha)$.



3 SUMMARY OF THE ELEMENTS AND FORMS ASSUMED BY THE GUMBEL DISTRIBUTION

Elements	Standard Shape	Y-shape
Random variable	z, with value $v = \frac{x - \mu}{s} \in R$	y, com value x ∈ R
Parameters	μ = 0, σ = 1	μ ∈ R, location parameter (mode), σ > 0, Dispersion Parameter
Amplitude	v ∈ ℝ	x ∈ ℝ
Notation	z ~Extremo(0.1)	and ~ Extreme(μ,σ)
Function probability density	**Maximum:** fz(v) = exp(-v-e-v) **Minimum:**	**Maximum:** fy(x) = 1σexp(-x-μσ-exp(-x-μσ)) **Minimal:**

	fz(v) = exp(v-exp(v))	fy(x) = 1σexp(x-μσ-exp(x-μσ))
Cumulative Distribution Function	**Maximum:** fz(v) = exp(-exp(-v)) **Minimal:** fz(v) = 1-exp(-exp(v))	**Maximum:** fy(x) = exp(-exp(-x-μσ)) **Minimal:** fy(x) = 1-exp(-exp(x-μσ))
Quantile function	**Maximum:** gz(p) = -ln(-ln(p)) p ∈ (0.1) **Minimal:** gz(p) = ln(-ln(1-p))	**Maximum:** gy(p) = μ - σln(-ln(p)) **Minimal:** gy(p) = μ + σln(-ln(1-p))
Average	**Maximum:** E(z) ≈ 0.5772 (constant Euler-Mascheroni) **Minimal:** E(z) ≈ -0.5772	**Maximum:** E(y) ≈ m + 0.5772σ **Minimal:** E(y) ≈ m - 0.5772σ
Variance	v(z) = π26	v(y) = σ2π26
Median	Ξz = -ln(ln(2)) ≈ 0.3665	Ξy = μ - σln(ln(2))
Fashion	0	M
Asymmetry coefficient	μ1 ≈ 1.14	μ1 ≈ 1.14
Kurosis coefficient	m2 = 125	m2 = 125
Moment Generator Function	**Maximum:** mz(t) = μ(1-t), t < 1 μ is the Gamma Function **Minimum:** mz(t) = μ(1+t), t > -1	**Maximum:** my(t) = exp(μt)μ(1-σt), t < 1σ **Minimum:** my(t) = exp(μt)μ(1+σt), t > -1σ
Characteristic Function	**Maximum:** ∅z(t) = μ(1-it) **Minimum:** ∅z(t) = μ(1+it)	**Maximum:** ∅y(t)=μ(1-iσt)exp(iμt) **Minimal:** ∅y(t) = μ(1+iσt)exp(iμt)

4 RELATIONS OF THE GUMBEL DISTRIBUTION TO OTHER DISTRIBUTIONS

1 If u ~ U(0,1) (standard uniform distribution), then y = μ - σln(-ln(u)) and y = μ + σln(-ln(1-u)) have, respectively, distribution from the extreme value to the maximum and to the minimum, with the parameter of location μ and dispersion σ;

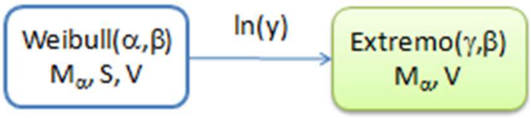
2 If x ~ Expo(1) (standard exponential destruction), then y = -ln(x) has distribution from the standard extreme value to the maximum and y = ln(x) has distribution from the standard extreme value to the minimum;

3 If y has distribution from the extreme value to the minimum, then x=ey has standard exponential distribution;

4 More generally:

- if x ~Weibull (α,λ) (Weibull distribution) with α shape parameter and dispersion λ, then y = ln(x) has distribution from extreme to minimum value with location parameter μ = ln(α) and dispersion σ = 1λ; and
- if y has distribution from extreme value to minimum with localization μ and dispersion σ then x = ey has Weibull distribution with format parameter 1σ and dispersion parameter eσ.

The figure below illustrates the relationships with the other districts (UFCG, 2017; USP, 2017; PA, 2017; CPRM, 2017; FRANCO *et all*, 2014; MOOD; GRAYBILL; BOES, 1974; MORETTIN; BUSSAB, 2010; TOMAZ, 2008; WATANABE, 2013; GIL, 2010; MARCONI; LAKATOS, 2008).



The Gumbel distribution is the minimum of the Normal and Log-normal distributions, and the maximum of the Exponential, Gamma, Normal, and Log-normal distributions (UFCG, 2017; USP, 2017; PA, 2017; CPRM, 2017; FRANCO *et all*, 2014; MOOD; GRAYBILL; BOES, 1974; MORETTIN; BUSSAB, 2010; TOMAZ, 2008; WATANABE, 2013), as shown in the following table:

Initial Distribution	**Limiting Distribution for Extremes**	
	Maximum	**Minimum**
Exponential	Gumbel	Weibull
Gamma	Gumbel	Weibull
Normal	Gumbel	Gumbel
Log-normal	Gumbel	Gumbel
Uniform	Weibull	Weibull
Pareto	Fréchet	Weibull
Cauchy	Fréchet	Fréchet

5 GUMBEL DISTRIBUTION ESTIMATORS (maximums)

MOM Method:

$$\bar{\alpha} = 0{,}7797\, s_x \quad \beta = \bar{x} - 0{,}45\, s_x$$

MVS Method:

α and β are the solutions of the following system of equations:

$$\frac{\partial}{\partial\alpha}\ln[L(\alpha,\beta)]=-\frac{N}{\alpha^1}+\frac{1}{\alpha^2}\sum_{i=1}^{N}(x_i-\beta)-\frac{1}{\alpha^2}\sum_{i=1}^{N}(x_i-\beta)\exp\left(\frac{-x_i-beta}{\alpha^1}\right)=0(F)$$

$$\frac{\partial}{\partial\alpha}\ln[L(\alpha,\beta)]=\frac{N}{\alpha^1}-\frac{1}{\alpha^1}\sum_{i=1}^{N}\exp\left(\frac{-x_i-beta}{\alpha^1}\right)=0(G)$$

By manipulating both equations, we get:

$$F(\alpha)=\sum_{i=1}^{N}x_i\exp\left(-\frac{x_i}{\alpha^1}\right)-\frac{1}{N}\sum_{i=1}^{N}(x_i-\alpha)\sum_{i=1}^{N}\exp\left(-\frac{x_i}{\alpha^1}\right)(H)$$

The solution of (H), by Newton's method, provides α .

Then $\overline{\beta}=\alpha*\ln\left[\frac{N}{\sum_{i=1}^{N}\exp\left(-\frac{x_i}{\alpha^1}\right)}\right]$

MML Method:

$$\overline{\alpha}=\frac{l_2}{\ln 2}\quad\beta=l_1-0{,}5772\,\overline{\alpha}$$

6 ESTIMATORS OF THE GUMBEL DISTRIBUTION (MINIMUM)

MOM Method:

$$\overline{\alpha}=0{,}7797\,s_x\quad\beta=\overline{x}-0{,}45\,s_x$$

MML Method:

$$\overline{\alpha}=\frac{l_2}{\ln 2}\quad\beta=l_1-0{,}5772\,\overline{\alpha}$$

7 CONCLUSIONS

The Gumbel Distribution is the probabilistic model used to model the distribution of the maximum. It appears as the limit of the maximum for a random sample of a statistical population with a standard exponential model.

By its nature, it is important in hydrology (to model river flows - monthly or annual maximums, for example), in finance, in modeling the behavior of earthquakes and other natural disasters. In addition, the probability distribution of the Generalized Extreme Value (GEV), using the ML method for the estimation of the parameters and the Gumbel distribution by the MV method, are considered, in the critical literature of the subject, the most appropriate for studies of the probability of maximum daily annual precipitation.

Due to the scarcity of articles that deal specifically with the specificities of the Gumbel Distribution, this work is very useful for its analysis and for the studies that make use of it.

REFERENCES

CPRM. COMPANHIA DE PESQUISA DE RECURSOS MINERAIS. Capítulo 6: Estimação de Parâmetros. Brasília-DF: CPRM, 2017. Disponível em <http://www.cprm. gov.br/publique/media/cap6-est_para.pdf>. Acessado em 11 de fevereiro de 2017.

FRANCO, Camila S. *et all*.. Distribuição de probabilidades para precipitação máxima diária na Bacia Hidrográfica do Rio Verde, Minas Gerais. Revista Brasileira de Engenharia Agrícola e Ambiental, vol. 18, n. 7, p. 735-741, 2014. 7 p. Disponível em <http://www.scielo. br/pdf/rbeaa/v18n7/v18n07a10.pdf>. Acessado em 11 de fevereiro de 2017.

GIL, Antônio Carlos. Como elaborar projetos de pesquisa. 5ª ed.. São Paulo: Atlas, 2010. 184 p.

MARCONI, Marina de Andrade; LAKATOS, Eva Maria. Técnicas de pesquisa: planejamento e execução de pesquisas, amostragens e técnicas de pesquisa, elaboração, análise e interpretação de dados. São Paulo: Atlas, 2008. 277 p.

MOOD, Alexander McFarlane; GRAYBILL, Franklin A.; BOES, Duane C.. Introduction to the theory of statistics. Nova Iorque-EUA: McGraw Hill, 1974. 577 p.

MORETTIN, Pedro A.; BUSSAB, Wilton de O.. Estatística Básica. 6a edição. Revista e atualizada. São Paulo: Saraiva, 2010. 557 p.

PA. PORTAL ACTION. Distribuição Gumbel (ou valor extremo). Disponível em <http://www.portalaction.com.br/probabilidades/614-distribuicao-gumbel-ou-valor-extremo>. Acessado em 11 de fevereiro de 2017.

TOMAZ, Plínio. Capítulo 151: Distribuição de Gumbel e Log-Pearson Tipo III. Karnataka-Índia: Kukke Subramanya Temple, 2008. Disponível em <http://www.pliniotomaz. com.br/downloads/Novos_livros/livro_calculoshidrolicos/capitulo151.pdf>. Acessado em 11 de fevereiro de 2017.

UFCG. UNIVERSIDADE FEDERAL DE CAMPINA GRANDE. Emil Julius Gumbel. Disponível em <http://www.dec.ufcg.edu.br/biografias/EmlJGumb.html>. Acessado em 11 de fevereiro de 2017.

USP. UNIVERSIDADE DE SÃO PAULO. Distribuição do valor extremo. Disponível em <http://www.galileu.esalq.usp.br/mostra_topico.php?cod=992>. Acessado em 11 de fevereiro de 2017.

WATANABE, Frederico Mamoru. Análise do Método de Gumbel para cálculo de vazões de dimensionamento de vertedouros. São Carlos-SP: EESC-USP, 2013. 89 p.